Universal separable metric space in the sense of isometry

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Abstract

The Banach space $C(2^\omega)$ under the uniform norm contains an isometric copy of every separable metric space.

Let (X, d) be a non-empty separable metric space. Fix an arbitrary point $p_0 \in X$. To each $p \in X$ we assign a function φ_p on X into $\mathbb{R}$ by

$$\varphi_p(x) = d(x, p) - d(x, p_0)$$

Then $\varphi_{p_0} \equiv 0$ and for every $p, q \in X$

$$\sup_{x \in X} |\varphi_p(x) - \varphi_q(x)| = d(p, q).$$

Note that each φ_p is bounded since $|\varphi_p(x)| \leq d(p, p_0)$. Let S be the linear subspace of the Banach space $C_0(X)$ of continous bounded functions on X under the supremum norm spanned by $\{\varphi_p\}_{p \in X}$. For $f \in S$, let

$$\|f\|_S = \sup_{x \in X} |f(x)|.$$

Let $\widehat{S}$ be the completion of $(S, \| \ \|_S)$. It is then a separable Banach space that contains an isometric copy of X. Since every separable Banach space is contained in $C(2^\omega)$ as an isomorphic copy (see below), we obtain

Theorem. *Every separable metric space is contained in $C(2^\omega)$ as an isometric image.*

We then show how every separable Banach space is contained in $C(2^\omega)$. Let $(E, \| \ \|_E)$ be a separable Banach space. Let $K = B_1(E^*)$ be the closed unit ball of the dual space $(E^*, \| \ \|^*)$ with the weak-$*$ topology. Under this topology K is compact and metrizable. For $p \in E$ define a function f_p on K by $f_p(\lambda) = (p, \lambda)$. Then $f_p \in C(K)$ and

$$\begin{aligned} \|f_p - f_q\|_{C(K)} &= \sup_{\lambda \in K} |(p, \lambda) - (q, \lambda)| \\ &= \sup_{\lambda \in K} |(p - q, \lambda)| \\ &= \|p - q\|_E. \end{aligned}$$

Therefore the mapping $p \mapsto f_p$ is an isometric embedding of E into $C(K)$. Since K is a compact metrizable space, there is a continumous mapping Π on 2^ω onto K. Define $\Phi : C(K) \to C(2^\omega)$ by $\Phi(f) = f \circ \Pi$. This Φ is an isometric embedding of $C(K)$ into $C(2^\omega)$. In this way we can embed E into $C(2^\omega)$.