Remarks on Erdős-Sierpiński duality

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meager and null ideals

Let X be a topological space.

$S \subset X$ is *nowhere dense* $\overset{\text{def}}{\Leftrightarrow} \mathrm{Int}(\mathrm{Cl}(S)) = \emptyset$.

$S \subset X$ is *meager* $\overset{\text{def}}{\Leftrightarrow} S = \bigcup_{n\in\omega} S_n$, with S_n nowhere dense.

The *meager ideal:*

$$\mathcal{M} \overset{\text{def}}{=} \text{the family of all meager subsets of } \mathbb{R}.$$

meager and null ideals

The Baire Category Theorem

In any complete metric space, nonempty open sets are never meager.

The meager ideal $\mathcal{M}$ is a *σ-ideal:*

1. $\emptyset \in \mathcal{M}$,
2. $\mathbb{R} \notin \mathcal{M}$,
3. if $A \subset B \in \mathcal{M}$ then $A \in \mathcal{M}$,
4. if $A_n \in \mathcal{M}$ $(\forall n \in \omega)$ then $\bigcup_{n\in\omega} A_n \in \mathcal{M}$.

So $\mathcal{M}$ gives a notion of *smallness* of subsets of $\mathbb{R}$. The Baire Category Theorem tells that nonempty open sets are not small in this sense.

meager and null ideals

The *Lebesgue outer measure* of $S \subset \mathbb{R}$:

$$\mu^*(S) = \inf \left\{ \sum_{n \in \omega} \mathrm{length}(I_n) \,\middle|\, S \subset \bigcup_{n \in \omega} I_n \right\}$$

$S \subset \mathbb{R}$ is *null* if $\mu^*(S) = 0$ (i.e., has Lebesgue measure 0).

The *null ideal:*

$$\mathcal{N} \stackrel{\mathrm{def}}{=} \text{the family of all null subsets of } \mathbb{R}.$$

meager and null ideals

The null ideal $\mathcal{N}$ is a σ-ideal:

1. $\emptyset \in \mathcal{N}$,
2. $\mathbb{R} \notin \mathcal{N}$,
3. if $A \subset B \in \mathcal{N}$ then $A \in \mathcal{N}$,
4. if $A_n \in \mathcal{N}$ $(\forall n \in \omega)$ then $\bigcup_{n\in\omega} A_n \in \mathcal{N}$.

So $\mathcal{N}$ gives *another* notion of smallness of subsets of $\mathbb{R}$. Nonempty open sets are not small in this sense either.

meager and null ideals

The ideals $\mathcal{M}$ and $\mathcal{N}$ are *Borel supported:*

- ⋆ If $A \in \mathcal{M}$ then $\exists B$, an F_σ set, $A \subset B \in \mathcal{M}$.
- ⋆ If $A \in \mathcal{N}$ then $\exists B$, a G_δ set, $A \subset B \in \mathcal{N}$.

So far, $\mathcal{M}$ and $\mathcal{N}$ look very similar.
This *similarity of $\mathcal{M}$ and $\mathcal{N}$* is the very theme of today's talk.

meager and null ideals

Moreover, $\mathcal{M}$ and $\mathcal{N}$ are *complementary:* there is a decomposition

$$\mathbb{R} = A \cup B, \quad A \in \mathcal{M}, \quad B \in \mathcal{N}.$$

So A is small in the sense of $\mathcal{M}$ while it holds almost everything in the sense of $\mathcal{N}$.

(Example) Let B be the set of all *Liouville numbers:* irrational α such that for every $n \geq 1$ there are pairs of integers p, q satisfying

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^n},$$

then put $A = \mathbb{R} \setminus B$.

continuum hypothesis

ω_1 is the smallest uncountable ordinal
$=$ the order-type of the set of all countable ordinals.

$\aleph_1$ is the cardinal of ω_1.

G.Cantor's *continuum hypothesis* (CH) is the assertion

$$2^{\aleph_0} = \aleph_1.$$

continuum hypothesis

So, CH states that the whole $\mathbb{R}$ can be indexed by the countable ordinals, i.e., all real numbers are enumerated like

$$x_0, x_1, \ldots, x_\alpha, \ldots \quad (\alpha < \omega_1),$$

so that there are only countably many reals before each x_α.

ZFC proves that there are only $2^{\aleph_0}$ Borel sets.

CH implies that all Borels sets are indexed by the countable ordinals. (This is the gimmick for our constructions.)

continuum hypothesis

Alert!

In the rest of this talk, we assume CH throughout.

Recall that CH is *consistent with* conventional ZFC axioms of set theory.

Every consequence of CH is also consistent with ZFC.

Erdős-Sierpiński duality

The Erdős-Sierpiński Duality Theorem (ZFC+CH)

There exists a mapping $f : \mathbb{R} \to \mathbb{R}$ such that

1. $f \circ f = \mathrm{id}_{\mathbb{R}}$,
2. $S \in \mathcal{M} \iff f(S) \in \mathcal{N} \quad (\forall S \subset \mathbb{R})$.

Let us call such a bijection an *Erdős-Sierpiński involution*.

Erdős-Sierpiński duality

Lemma

If $A \in \mathcal{M}$ or $A \in \mathcal{N}$ then there is K such that

$$K \in \mathcal{M} \cap \mathcal{N}, \quad A \cap K = \emptyset, \quad K \approx 2^\omega$$

where 2^ω is the Cantor space.

Sketch: $\mathbb{R} \setminus A$ contains an uncountable Borel set. So by the Alexandroff-Hausdorff theorem $\mathbb{R} \setminus A$ contains a homeomorphic copy C of the Cantor space 2^ω. C is nowhere dense in $\mathbb{R}$. So $C \in \mathcal{M}$. By the "split and shrink" argument, we can extract null subset K of C, still homeomorphic to 2^ω.

Erdős-Sierpiński duality

How to construct an ES involution:

Let us enumerate all meager F_σ sets and all null G_δ sets like

$$\begin{aligned} &A_0, A_1, \ldots, A_\alpha, \ldots \quad (\alpha < \omega_1), \\ &B_0, B_1, \ldots, B_\alpha, \ldots \quad (\alpha < \omega_1). \end{aligned}$$

We may assume $A_0 = \mathbb{R} \setminus B_0$.

Erdős-Sierpiński duality

For $\alpha < \omega_1$, we recursively choose sets P_α and Q_α.

First we put $P_0 = A_0$ and $Q_0 = B_0$.

Having obtained, P_ξ, Q_ξ $(\xi < \alpha)$, suppose all P_ξ's are meager and Q_ξ's are null. Find (by the Lemma) K_α and L_α such that

- $K_\alpha \approx L_\alpha \approx 2^\omega$.
- $\bigcup_{\xi<\alpha} P_\xi \cap K_\alpha = \bigcup_{\xi<\alpha} Q_\xi \cap L_\alpha = \emptyset$.
- $K_\alpha, L_\alpha \in \mathcal{M} \cap \mathcal{N}$.

Erdős-Sierpiński duality

The sets

$$A_\alpha, \quad K_\alpha, \quad \bigcup_{\xi<\alpha} P_\xi$$

are meager. So we can find a meager F_σ set P_α such that

$$P_\alpha \supset A_\alpha \cup K_\alpha \cup \bigcup_{\xi<\alpha} P_\xi.$$

Similarly we can find a null G_δ set Q_α such that

$$Q_\alpha \supset B_\alpha \cup L_\alpha \cup \bigcup_{\xi<\alpha} Q_\xi.$$

Erdős-Sierpiński duality

Thus we have obtained two increasing ω_1-sequeces

$$P_0 \subset P_1 \subset \cdots \subset P_\alpha \subset \cdots \quad (\alpha < \omega_1),$$
$$Q_0 \subset Q_1 \subset \cdots \subset Q_\alpha \subset \cdots \quad (\alpha < \omega_1)$$

of Borel sets such that

$$S \in \mathcal{M} \iff \exists\alpha < \omega_1(S \subset P_\alpha),$$
$$S \in \mathcal{N} \iff \exists\alpha < \omega_1(S \subset Q_\alpha).$$

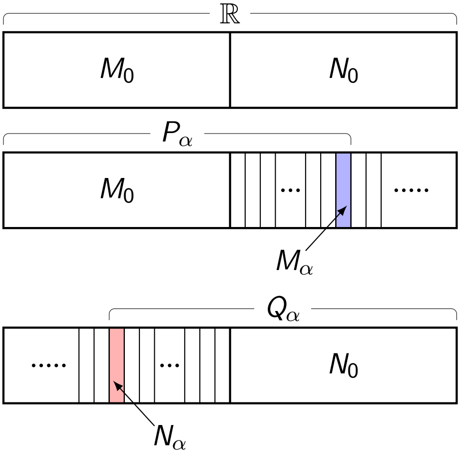
For every α, put

$$M_\alpha \overset{\text{def}}{=} P_\alpha \setminus \bigcup_{\xi<\alpha} P_\xi, \quad N_\alpha \overset{\text{def}}{=} Q_\alpha \setminus \bigcup_{\xi<\alpha} Q_\xi$$

They both have size $2^{\aleph_0}$. ($\because K_\alpha \subset M_\alpha$, $L_\alpha \subset N_\alpha$).

Erdős-Sierpiński duality

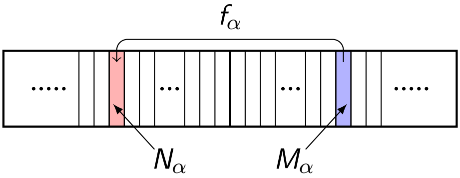
The sets M_α and N_α give rise to complementary pair of partitions:



Erdős-Sierpiński duality

Now for each nonzero $\alpha < \omega_1$ choose a bijection

$$f_\alpha : M_\alpha \xrightarrow[\text{onto}]{1-1} N_\alpha$$



and then put

$$f(x) = \begin{cases} f_\alpha(x), & \text{if } x \in M_\alpha,\ \alpha \geq 1, \\ f_\alpha^{-1}(x), & \text{if } x \in N_\alpha,\ \alpha \geq 1. \end{cases}$$

Erdős-Sierpiński duality

By the construction,

$$f \circ f = \mathrm{id}_{\mathbb{R}} \quad \text{and}$$
$$f(P_\alpha) = Q_\alpha \quad (\forall \alpha < \omega_1).$$

If $S \in \mathcal{M}$ then $S \subset P_\alpha$ for some $\alpha < \omega_1$.

Then $f(S) \subset f(P_\alpha) = Q_\alpha$, and so $f(S) \in \mathcal{N}$.

Similarly, if $S \in \mathcal{N}$ then $f(S) \in \mathcal{M}$.

ES involutions are not so well-behaved

ES involutions are not so well-behaved.

Fact

Every Erdős-Sierpiński involution is *Lebesgue non-measurable*.

Proof: Suppose an ES involution f is measurable. By Lusin's theorem there is a compact set K such that $\mu(K) > 0$ and $f \restriction K$ is continuous. We can always take K nowhere dense. $f(K)$, a compact non-meager set, must contain an interval. Since $f(K) \approx K$, it follows that K must contain an interval. Contradiction!

Similarly, ES involutions are *BP non-measurable*; i.e., there is no dense G_δ set D such that $f \restriction D$ is continous on D.

Some ES involutions behave better

Since our M_α and N_α ($\alpha \geq 1$) are uncountable Borel sets, we can choose f_α to be a *Borel isomorphism*.

If f is so-constructed ES involution, then for every $S \in \mathcal{M} \cap \mathcal{N}$, $f \restriction S$ is a Borel isomorphism of S onto $f(S)$. Consequently we have

Theorem 1a (ZFC+CH)

There is an Erdős-Sierpiński involution which is *Marczewski measurable*.

By definition, f is Marczewski measurable iff for every perfect set K there is a perfect set $K' \subset K$ such that $f \restriction K'$ is continous on K'.

Some ES involutions behave worse

On the other hand,

Theorem 1b (ZFC+CH)

There is an Erdős-Sierpiński involution which is *Marczewski non-measurable*.

Proof: Let $X_\alpha \subset M_\alpha$ be a Bernstein set relative to M_α and let f_α map X_α onto an open set relative to N_α. If $C \subset \mathbb{R}$ is perfect, then $C \cap M_\alpha$ contains a perfect set for some non-zero $\alpha < \omega_1$. Let $K \subset C \cap M_\alpha$ be *any* perfect set. Then $f \restriction K$ is not continous because a non-Borel set $K \cap X_\alpha$ is the pre-image of an open set.

the Hilbert cube

The Hilbert cube $\mathbb{H}$ is the infinite product $[0,1]^{\aleph_0}$:

$$\mathbb{H} = [0,1] \times [0,1] \times [0,1] \times \cdots$$

Give $\mathbb{H}$ the product topology and the product measure.

We can talk about $\mathcal{M}$ and $\mathcal{N}$ on $\mathbb{H}$.
There are ES involutions $f : \mathbb{H} \to \mathbb{H}$.

dimension theory

Fact (ZFC)

The Hilbert cube $\mathbb{H}$ cannot be covered by a countable family of finite-dimensional sets.

Fact (ZFC)

The Hilbert cube $\mathbb{H}$ can be covered by $\aleph_1$ many zero-dimensional sets.

Fact (ZFC)

Every finite-dimensional subset of $\mathbb{H}$ is contained in a G_δ set of the same dimension.

dimension theory

Let $\mathcal{D}$ be the family of $S \subset \mathbb{H}$ which is *countable dimensional*; i.e., covered by a countable family of finite-dimensional sets.

Then $\mathcal{D}$ is a σ-ideal:

1. $\emptyset \in \mathcal{D}$,
2. $\mathbb{H} \notin \mathcal{D}$,
3. if $A \subset B \in \mathcal{D}$ then $A \in \mathcal{D}$,
4. if $A_n \in \mathcal{D}$ $(\forall n \in \omega)$ then $\bigcup_{n\in\omega} A_n \in \mathcal{D}$.

Moreover, for every $S \in \mathcal{D}$ there is an $\mathrm{G}_{\delta\sigma}$ set $D \in \mathcal{D}$ such that $S \subset D$; i.e., $\mathcal{D}$ is Borel supported.

$\mathcal{D}$ looks quite similar to $\mathcal{M}$ and $\mathcal{N}$.

dimension theory

Set theorists do not pay much attention to $\mathcal{D}$ because ZFC proves

$$\mathrm{non}(\mathcal{D}) = 2^{\aleph_0} \text{ and } \mathrm{cov}(\mathcal{D}) = \aleph_1.$$

(For $\mathcal{M}$ or $\mathcal{N}$, ZFC does not prove anything like this.)

the third party

Define $D_0 \subset \mathbb{H}$ by

$$D_0 = \{ x \in \mathbb{H} \mid \forall n \in \omega (x(n) \notin \mathbb{Q}) \} = ([0,1] \setminus \mathbb{Q})^{\aleph_0}.$$

D_0 is zero-dimensional. So $D_0 \in \mathcal{D}$. D_0 is negligible in the sense of $\mathcal{D}$.

Let $C_0 = \mathbb{H} \setminus D_0$.

$C_0 \in \mathcal{M} \cap \mathcal{N}$, since D_0 is a dense G_δ set of measure 1.

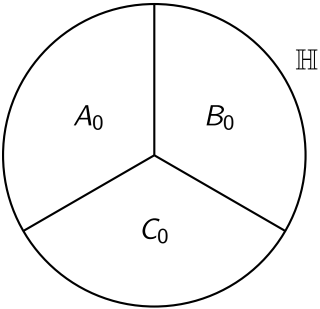
the third party

There are sets A_0 and B_0 such that

$$D_0 = A_0 \cup B_0, \quad A_0 \in \mathcal{M}, \quad B_0 \in \mathcal{N}$$

For example, let A_0 be the set of all $x \in D_0$ such that $x(n)$ is not a Liouville number for some n, and then put $B_0 = D_0 \setminus A_0$.

the third party



$$A_0, C_0 \in \mathcal{M}$$
$$B_0, C_0 \in \mathcal{N}$$
$$A_0, B_0 \in \mathcal{D}$$

$$P_0 \stackrel{\text{def}}{=} A_0 \cup C_0,$$
$$Q_0 \stackrel{\text{def}}{=} B_0 \cup C_0,$$
$$R_0 \stackrel{\text{def}}{=} A_0 \cup B_0.$$

Then we choose three ω_1-sequences of Borel sets

$$\begin{array}{l} P_0 \subset P_1 \subset \cdots \subset P_\alpha \subset \cdots \\ Q_0 \subset Q_1 \subset \cdots \subset Q_\alpha \subset \cdots \quad (\alpha < \omega_1). \\ R_0 \subset R_1 \subset \cdots \subset R_\alpha \subset \cdots \end{array}$$

the third party

We choose three ω_1-sequences of Borel sets

$$\begin{aligned} &P_0 \subset P_1 \subset \cdots \subset P_\alpha \subset \cdots \\ &Q_0 \subset Q_1 \subset \cdots \subset Q_\alpha \subset \cdots \quad (\alpha < \omega_1) \\ &R_0 \subset R_1 \subset \cdots \subset R_\alpha \subset \cdots \end{aligned}$$

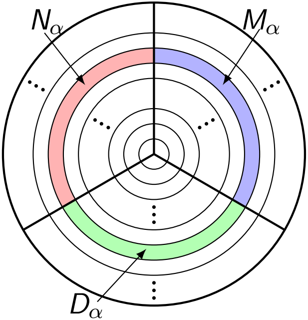
so that

- $P_\alpha \in \mathcal{M}$, $Q_\alpha \in \mathcal{N}$, $R_\alpha \in \mathcal{D}$,
- $\bigcup_{\alpha<\omega_1} P_\alpha = \bigcup_{\alpha<\omega_1} Q_\alpha = \bigcup_{\alpha<\omega_1} R_\alpha = \mathbb{R}$,
- $P_\alpha \setminus \bigcup_{\xi<\alpha} P_\xi$, $Q_\alpha \setminus \bigcup_{\xi<\alpha} Q_\xi$, $R_\alpha \setminus \bigcup_{\xi<\alpha} R_\xi$ are all uncountable.

the third party

For nonzero $\alpha < \omega_1$, put

$$\begin{cases} M_\alpha = P_\alpha \setminus \bigcup_{\xi<\alpha} P_\xi \\ N_\alpha = Q_\alpha \setminus \bigcup_{\xi<\alpha} Q_\xi \\ D_\alpha = R_\alpha \setminus \bigcup_{\xi<\alpha} R_\xi \end{cases}$$



They are uncountable Borel sets. So they are Borel isomorphic.

first variation

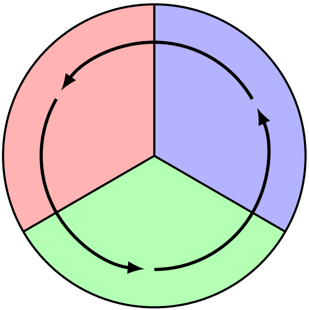
For each nonzero α, choose Borel isomorphisms

$$f_\alpha \colon M_\alpha \to N_\alpha, \quad g_\alpha \colon N_\alpha \to D_\alpha.$$

Then put for each $x \in \mathbb{H}$,

$$f(x) = \begin{cases} f_\alpha(x), & \text{if } x \in M_\alpha,\ \alpha \geq 1, \\ g_\alpha(x), & \text{if } x \in N_\alpha,\ \alpha \geq 1, \\ (g_\alpha \circ f_\alpha)^{-1}(x), & \text{if } x \in D_\alpha,\ \alpha \geq 1. \end{cases}$$

first variation



We obtain a bijection $f\colon \mathbb{H} \to \mathbb{H}$ such that

- $f \circ f \circ f = \mathrm{id}_{\mathbb{R}}$,
- For every $S \subset \mathbb{H}$, $S \in \mathcal{M}$ iff $f(S) \in \mathcal{N}$ iff $f(f(S)) \in \mathcal{D}$.

first variation

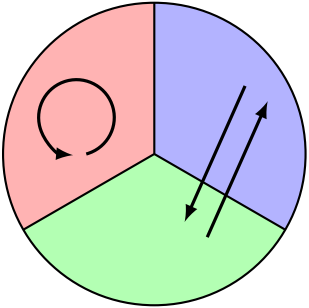
Thus we obtain

Theorem 2 (ZFC+CH)

There is a bijection $f \colon \mathbb{H} \to \mathbb{H}$ such that

- $f \circ f \circ f = \mathrm{id}_{\mathbb{R}}$,
- $S \subset \mathbb{H}$ is meager iff $f(S)$ is null iff $f(f(S))$ is countable-dimensional,
- if S is a countable-dimensional null closed set, then $f \restriction S$ is Borel measurable. ($\therefore$ f is Marczewski measurable)

second variation



Let us define another bijection $g\colon \mathbb{H} \to \mathbb{H}$ by

$$g(x) = \begin{cases} (g_\alpha \circ f_\alpha)(x), & \text{if } x \in M_\alpha,\ \alpha \geq 1, \\ x, & \text{if } x \in N_\alpha,\ \alpha \geq 1, \\ (g_\alpha \circ f_\alpha)^{-1}(x), & \text{if } x \in D_\alpha,\ \alpha \geq 1. \end{cases}$$

second variation

Now we have $g \circ g = \mathrm{id}_{\mathbb{R}}$ and $S \in \mathcal{M}$ iff $g(S) \in \mathcal{D}$. Moreover, $g(x) = x$ for almost every x (in the sense of $\mathcal{N}$). Therefore,

Theorem 3 (ZFC+CH)

There is a bijection $g \colon \mathbb{H} \to \mathbb{H}$ such that

- $g \circ g = \mathrm{id}_{\mathbb{R}}$,
- $S \subset \mathbb{H}$ is meager iff $g(S)$ is countable-dimensional,
- if S is countable-dimensional closed set, then $g \restriction S$ is Borel measurable ($\therefore$ g is Marczewski measurable),
- g is Lebesgue measurable.